Exercise 5

The table gives estimates of the world population, in millions, from 1750 to 2000.

Year	Population	Year	Population
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.

(b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.

(c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

Solution

Part (a)

Use an exponential model for the population.

$$P(t) = P_0 e^{kt}$$

Use the populations at 1750 and 1800 to construct a system of equations for the two unknowns, P_0 and k.

$$\begin{cases} P(1750) = P_0 e^{k(1750)} = 790 \\ P(1800) = P_0 e^{k(1800)} = 980 \end{cases}$$

Divide both sides of the second equation by those of the first equation to eliminate P_0.

$$\frac{P_0 e^{k(1800)}}{P_0 e^{k(1750)}} = \frac{980}{790}$$

$$e^{50k} = \frac{98}{79}$$

$$\ln e^{50k} = \ln \frac{98}{79}$$

$$50k = \ln \frac{98}{79}$$

$$k = \frac{1}{50} \ln \frac{98}{79} \approx 0.00431039 \text{ year}^{-1}$$

Substitute this formula for k into either of the two equations to get P_0.

$$P_0 e^{k(1750)} = 790$$

$$P_0 e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)(1750)} = 790$$

$$P_0 = \frac{790}{e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)(1750)}} \approx 0.418468 \text{ million}$$

Therefore, the population model using the populations at 1750 and 1800 is

$$P(t) = \left[\frac{790}{e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)(1750)}}\right] e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)t}.$$

The world populations in 1900 and 1950 are, respectively,

$$\begin{cases} P(1900) = \left[\dfrac{790}{e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)(1750)}}\right] e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)1900} \approx 1508.08 \text{ million} \\ P(1950) = \left[\dfrac{790}{e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)(1750)}}\right] e^{\left(\frac{1}{50}\ln\frac{98}{79}\right)1950} \approx 1870.78 \text{ million} \end{cases}.$$

Use the percent difference formula to see how far off these numbers are from the actual values.

$$\begin{cases} 1900\text{:} & \dfrac{1508.08 - 1650}{1650} \times 100\% \approx -8.60128\% \\ 1950\text{:} & \dfrac{1870.78 - 2560}{2560} \times 100\% \approx -26.9226\% \end{cases}$$

The model's population in 1900 underestimates the actual value by about 8.60%, and the model's population in 1950 underestimates the actual value by about 26.9%.

Part (b)

Use an exponential model for the population.

$$P(t) = P_0 e^{kt}$$

Use the populations at 1850 and 1900 to construct a system of equations for the two unknowns, P_0 and k.

$$\begin{cases} P(1850) = P_0 e^{k(1850)} = 1260 \\ P(1900) = P_0 e^{k(1900)} = 1650 \end{cases}$$

Divide both sides of the second equation by those of the first equation to eliminate P_0.

$$\frac{P_0 e^{k(1900)}}{P_0 e^{k(1850)}} = \frac{1650}{1260}$$

$$e^{50k} = \frac{55}{42}$$

$$\ln e^{50k} = \ln \frac{55}{42}$$

$$50k = \ln \frac{55}{42}$$

$$k = \frac{1}{50} \ln \frac{55}{42} \approx 0.00539327 \text{ year}^{-1}$$

Substitute this formula for k into either of the two equations to get P_0.

$$P_0 e^{k(1850)} = 1260$$

$$P_0 e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)(1850)} = 1260$$

$$P_0 = \frac{1260}{e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)(1850)}} \approx 0.0585025 \text{ million}$$

Therefore, the population model using the populations at 1850 and 1900 is

$$P(t) = \left[\frac{1260}{e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)(1850)}}\right] e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)t}.$$

The world populations in 1900 and 1950 are, respectively,

$$\begin{cases} P(1900) = \left[\frac{1260}{e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)(1850)}}\right] e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)1900} = 1650 \text{ million} \\ P(1950) = \left[\frac{1260}{e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)(1850)}}\right] e^{\left(\frac{1}{50} \ln \frac{55}{42}\right)1950} \approx 2160.71 \text{ million} \end{cases}.$$

Use the percent difference formula to see how far off these numbers are from the actual values.

$$\begin{cases} 1900: \quad \frac{1650 - 1650}{1650} \times 100\% = 0\% \\ 1950: \quad \frac{2160.71 - 2560}{2560} \times 100\% \approx -15.5971\% \end{cases}$$

The model's population in 1900 exactly predicts the value because 1900 was one of the years used to formulate the model, and the model's population in 1950 underestimates the actual value by about 15.6%.

Part (c)

Use an exponential model for the population.

$$P(t) = P_0 e^{kt}$$

Use the populations at 1900 and 1950 to construct a system of equations for the two unknowns, P_0 and k.

$$\begin{cases} P(1900) = P_0 e^{k(1900)} = 1650 \\ \\ P(1950) = P_0 e^{k(1950)} = 2560 \end{cases}$$

Divide both sides of the second equation by those of the first equation to eliminate P_0.

$$\frac{P_0 e^{k(1950)}}{P_0 e^{k(1900)}} = \frac{2560}{1650}$$

$$e^{50k} = \frac{256}{165}$$

$$\ln e^{50k} = \ln \frac{256}{165}$$

$$50k = \ln \frac{256}{165}$$

$$k = \frac{1}{50} \ln \frac{256}{165} \approx 0.00878464 \text{ year}^{-1}$$

Substitute this formula for k into either of the two equations to get P_0.

$$P_0 e^{k(1900)} = 1650$$

$$P_0 e^{\left(\frac{1}{50} \ln \frac{256}{165}\right)(1900)} = 1650$$

$$P_0 = \frac{1650}{e^{\left(\frac{1}{50} \ln \frac{256}{165}\right)(1900)}} \approx 0.0000930583 \text{ million}$$

Therefore, the population model using the populations at 1900 and 1950 is

$$P(t) = \left[\frac{1650}{e^{\left(\frac{1}{50} \ln \frac{256}{165}\right)(1900)}}\right] e^{\left(\frac{1}{50} \ln \frac{256}{165}\right)t}.$$

The world population in 2000 is

$$P(2000) = \left[\frac{1650}{e^{\left(\frac{1}{50} \ln \frac{256}{165}\right)(1900)}}\right] e^{\left(\frac{1}{50} \ln \frac{256}{165}\right)2000} \approx 3971.88 \text{ million}.$$

Use the percent difference formula to see how far off this number is from the actual value.

$$2000: \quad \frac{3971.88 - 6080}{6080} \times 100\% \approx -34.673\%$$

The model's population in 2000 underestimates the actual value by about 34.7%. The reason the model is so bad is because it assumes that k is a constant for all time; in part (a) it was $0.00431039 \text{ year}^{-1}$, in part (b) it was $0.00539327 \text{ year}^{-1}$, and in part (c) it was $0.00878464 \text{ year}^{-1}$.